

# AN INTERTWINING OPERATOR FOR THE HARMONIC OSCILLATOR AND THE DIRAC OPERATOR WITH APPLICATION TO THE HEAT AND WAVE KERNELS

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**Abstract.** In this article an intertwining operator is constructed which transforms the harmonic oscillator to the Dirac operator (the first order derivative operator). We give also the explicit solutions to the heat and wave equation associated to Dirac operator. As an application the heat and the wave kernels of the harmonic oscillator are computed.

**Key words:** Intertwining Operator, Harmonic oscillator, Dirac operator, Heat equation, Wave equation, Fourier transform, confluent hypergeometric function

## 1 – Introduction

The harmonic oscillators are interesting and important in their own right and play a fundamental role in the modeling of the quantum fields and are related to many mathematical and physical problems ([1], [2], [6], [7], [8], [9]). The aim of this paper is to give an intertwining operator  $T$  which relate the harmonic oscillator:

$$L^a = \frac{\partial^2}{\partial x^2} - a^2 x^2 \quad a > 0 \quad (1.1)$$

to the Dirac operator (first order derivative operator):

$$D = \frac{\partial}{\partial X} \quad (1.2)$$

We give also the explicit solutions to the following heat and wave equations associated to Dirac operator:

$$(HD) \quad \begin{cases} \partial_t U(t, X) = \frac{\partial}{\partial X} U(t, X) & (t, X) \in R_+^* \times \mathbb{R} \\ U(0, X) = U_0(X) & U_0 \in C_0^\infty(\mathbb{R}) \end{cases}$$

$$(WD) \quad \begin{cases} \partial_t^2 V(t, X) = \frac{\partial}{\partial X} V(t, X) & (t, X) \in R_+^* \times \mathbb{R} \\ V(0, X) = 0 & \partial_t V(0, X) = V_0(X) \in C_0^\infty(\mathbb{R}) \end{cases}$$

Note that the Cauchy problem  $(HD)$  for the heat equation associated to the Dirac operator  $D$  is nothing but the Cauchy problem for the transport equation. The wave equation

(WD) associated to the Dirac operator is considered in [3] p.64 where the uniqueness of solution is obtained but there are no known explicit solutions until now.

As an application of our intertwining operator, from the heat and wave kernels of the Dirac operator, we give the explicit solutions of the following heat and wave equations associated to the harmonic oscillator:

$$\begin{aligned} (HL^a) \quad & \begin{cases} \partial_t u(t, x) = \left( \frac{\partial^2}{\partial x^2} - a^2 x^2 \right) u(t, x) & (t, x) \in \mathbb{R}_+^* \times \mathbb{R} \\ u(0, x) = u_0(x) & u_0 \in C_0^\infty(\mathbb{R}) \end{cases} \\ (WL^a) \quad & \begin{cases} \partial_t^2 v(t, x) = \left( \frac{\partial^2}{\partial x^2} - a^2 x^2 \right) v(t, x) & (t, x) \in \mathbb{R}_+^* \times \mathbb{R} \\ v(0, x) = 0 & \partial_t v(0, x) = v_0(x), v_0 \in \mathcal{S}_a(\mathbb{R}) \end{cases} \end{aligned}$$

with

$$\mathcal{S}_a(\mathbb{R}) = \left\{ \phi : \mathcal{F} \left[ e^{\frac{-ax^2}{2}} \phi \right] (\xi) \in C_0^\infty(\mathbb{R}) \right\}$$

Note that the heat kernel for the harmonic oscillator is known for long time [1]:

$$K_a(x, x', t) =$$

$$\sqrt{\frac{a}{2\pi}} \frac{1}{\sqrt{\sinh(2at)}} \exp \left[ -\frac{a}{2}(x^2 + x'^2) \coth(2at) + \frac{axx'}{\sinh(2at)} \right] \quad (1.3)$$

but to our knowledge the method used here and the obtained formula are new.

For the wave kernel of the harmonic oscillator  $w_a(t, x, x')$ , the following integral representation is given for the wave kernel at the origin in [2]p.358:

$$w_1(x, 0, t) = \frac{i}{4\pi} \int_C \sqrt{\frac{1}{2z \sinh(z)}} e^{\frac{t^2}{2z} - \frac{x^2 \coth(z)}{2}} dz \quad (1.4)$$

where  $C$  is a contour symmetric with respect to the  $x$ -axis going through the origin obtained by a smooth deformation of the circle  $C(\frac{1}{2}c, \frac{1}{2}c)$  of center  $c/2$  and radius  $c/2$  and for  $t < |x|$  the kernel vanishes.

Now we recall some facts about the Fourier transform:

for  $f \in L^1(\mathbb{R})$  the Fourier transform of  $f$  and its inverse:

$$(\mathcal{F}f)(\xi) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{-ix\xi} f(x) dx \quad (1.5)$$

$$(\mathcal{F}^{-1}f)(x) = \frac{1}{\sqrt{2\pi}} \int_{-\infty}^{+\infty} e^{ix\xi} f(\xi) d\xi \quad (1.6)$$

and we have the following formulas:

$$\mathcal{F}f(\alpha\xi) = \frac{1}{\alpha} \mathcal{F}f\left(\frac{x}{\alpha}\right)(\xi) \quad (1.7)$$

$$\mathcal{F}^{-1}[e^{-s\xi^2}](x) = \frac{1}{\sqrt{2s}} e^{-x^2/4s}, \quad s > 0 \quad (1.8)$$

Recall also the complementary error function [3] p.272

$$\operatorname{Erfc}(z) = \int_z^\infty e^{-t^2} dt \quad (1.9)$$

and in terms of the Tricomi confluent hypergeometric  $U(a, c, z)$ :

$$\operatorname{Erfc}(z) = \frac{1}{2} z e^{-z^2} U(1, 3/2, z^2) = \frac{1}{2} e^{-z^2} U(1/2, 1/2, z^2) \quad (1.10)$$

## 2-The intertwining operator

**Definition 2.1** An operator  $T$  is said to be an intertwining operator if it relates operators,  $L$  and  $D$ , by

$$TL = DT \quad (2.1)$$

**Lemma 2.2** For  $a > 0$  and  $\phi \in \mathcal{S}_a(\mathbb{R})$  we have

$$\left[ \frac{\partial^2}{\partial x^2} - a^2 x^2 \right] \phi(x) = e^{\frac{ax^2}{2}} \mathcal{F}^{-1} \left[ e^{-\frac{\xi^2}{4a} - \frac{\log|\xi|}{2}} (-2a\xi \frac{\partial}{\partial \xi}) e^{\frac{\xi^2}{4a} + \frac{\log|\xi|}{2}} \mathcal{F} \left[ e^{-\frac{ax^2}{2}} \phi \right] (\xi) \right] (x) \quad (2.2)$$

**Proof:** Set

$$\phi(x) = e^{\frac{ax^2}{2}} \psi(x) \quad (2.3)$$

we get

$$e^{\frac{ax^2}{2}} \left[ \frac{\partial^2}{\partial x^2} - a^2 x^2 \right] e^{\frac{ax^2}{2}} \psi(x) = \left[ \frac{\partial^2}{\partial x^2} + 2ax \frac{\partial}{\partial x} + a \right] \psi(x) \quad (2.4)$$

take the Fourier transform

$$\mathcal{F} \left[ e^{\frac{ax^2}{2}} \left[ \frac{\partial^2}{\partial x^2} - a^2 x^2 \right] e^{\frac{ax^2}{2}} \psi(x) \right] (\xi) = \left[ -\xi^2 - 2a\xi \frac{\partial}{\partial \xi} - a \right] (\mathcal{F}\psi)(\xi) \quad (2.5)$$

set

$$(\mathcal{F}\psi)(\xi) = e^{-\frac{\xi^2}{4a} - \frac{\log|\xi|}{2}} w(\xi) \quad (2.6)$$

we obtain

$$e^{\frac{\xi^2}{4a} + \frac{\log|\xi|}{2}} \mathcal{F} \left[ e^{-\frac{ax^2}{2}} \left[ \frac{\partial^2}{\partial x^2} - a^2 x^2 \right] e^{\frac{ax^2}{2}} \psi(x) \right] (\xi) = -2a\xi \frac{\partial w(\xi)}{\partial \xi} \quad (2.7)$$

using (2.6)) we have

$$-2a\xi \frac{\partial w(\xi)}{\partial \xi} = e^{\frac{\xi^2}{4a} + \frac{\log|\xi|}{2}} \mathcal{F} \left[ e^{-\frac{ax^2}{2}} \left[ \frac{\partial^2}{\partial x^2} - a^2 x^2 \right] e^{\frac{ax^2}{2}} \mathcal{F}^{-1} [e^{-\frac{\xi^2}{4a} - \frac{\log|\xi|}{2}} w(\xi)](x) \right] (\xi) \quad (2.8)$$

**Proposition 2.3** If  $f \in \mathcal{S}(\mathbb{R})$  and set

$$(T\phi)(\xi)_{|\xi|=\exp(-2aX)} = e^{\frac{\xi^2}{4a} + \frac{\log|\xi|}{2}} \mathcal{F} \left[ e^{-\frac{ax^2}{2}} \phi \right] (\xi)_{|\xi|=\exp(-2aX)} \quad (2.9)$$

Then the operator  $T$  is an intertwining operator for the harmonic oscillator  $L^a$  given in (1.1) and the Dirac operator  $D$  in (1.2):

$$TL^a = DT \quad (2.10)$$

**Proof** The formula (2.10) is a consequence of the lemma 2.2

### 3-Heat and wave kernels for the Dirac operator

**Proposition 3.1** The Cauchy problem  $(HD)$  for the heat equation associated to the Dirac operator has the unique solution given by

$$U(t, X) = U_0(t + X) \quad (3.1)$$

**Proof:** The Cauchy problem for the heat equation associated to the Dirac operator  $D$  reduced to the Cauchy problem for the classical transport equation and its unique solution is given by (3.1).

**Proposition 3.2** The function

$$W(t, X, X') = \frac{t}{\sqrt{4\pi|X - X'|}} \exp\left(\frac{-t^2}{4|X - X'|}\right) U\left(1, \frac{3}{2}, \frac{t^2}{4|X - X'|}\right) \quad (3.2)$$

satisfies the wave equation associated Dirac operator  $D$ .  
and we have

$$W_a(t, X, X') = \frac{2}{\sqrt{\pi}} \operatorname{Erfc}\left(\frac{t}{\sqrt{4|X - X'|}}\right) \quad (3.2)'$$

where  $\operatorname{Erfc}(x)$  is the complementary error function in (1.9).

**Proof:** Set  $z = -\frac{t^2}{4(X-X')}$  we get  $\frac{\partial}{\partial t} = -\frac{2t}{4(X-X')} \frac{\partial}{\partial z}$ ,  
 $\frac{\partial^2}{\partial t^2} = \frac{t^2}{4(X-X')^2} \frac{\partial^2}{\partial z^2} - \frac{2}{4(X-X')} \frac{\partial}{\partial z}$ ,  $\frac{\partial}{\partial X} = \frac{t^2}{4(X-X')^2} \frac{\partial}{\partial z}$ .

$$\frac{\partial}{\partial X} \phi(t, X) - \frac{\partial^2}{\partial t^2} \phi(t, X) = \frac{1}{X - X'} \left\{ z \frac{\partial}{\partial z^2} + (1/2 - z) \frac{\partial}{\partial z} \right\} = 0 \quad (3.3)$$

the wave equation associated to the first order derivative operator is equivalent to the equation of confluent hypergeometric type [5] p.268

$$z\varphi''(z) + (1/2 - z)\varphi'(z) = 0 \quad (3.4)$$

and an appropriate solution [5] p.270 is  $z^{1/2} \exp(z) U(1, 3/2, -z)$ .

**Theorem 3.3** The Cauchy problem  $(WD)$  for the wave equation associated to the Dirac operator has the unique solution given by:

$$V(t, X) = \int_{|X-X'| < \frac{t}{2}} W(X, X') V_0(X') dX' \quad (3.5)$$

where  $W(t, X, X')$  is given by (3.2) and (3.2)'.

**Proof** In view of the proposition 3.2 it remain to show the limit conditions in  $(WD)$ . And for this set  $X' = X + \frac{t}{2}s$  we have

$$V(t, X) = Ct^{3/2} \int_{-1}^1 \frac{V_0(X + s\frac{t}{2})}{\sqrt{|s|}} \exp\left(-\frac{t}{2|s|}\right) U\left(1, 3/2, \frac{t}{2|s|}\right) ds \quad (3.6)$$

where  $C = 2^{-1/2} \frac{1}{\sqrt{4\pi}}$ . By the formula giving the first derivative of the Lommel confluent hypergeometric function [4] p.265

$$\frac{d}{dz} U(a, c, z) = -aU(a+1, c+1, z) \quad (3.7)$$

we obtain the limit conditions using the following behavior of the degenerate confluent hypergeometric function  $U(a, c, z)$ , [5] p.288 – 289 For  $z \rightarrow 0$ :

$$U(a, c, z) = (\Gamma(c-1)/\Gamma(a))z^{1-c} + O(1), 1 < \Re c < 2 \quad (3.8)$$

$$U(a, c, z) = (\Gamma(c-1)/\Gamma(a))z^{1-c} + O(|z|^{\Re c-2}), \Re c \geq 2, c \neq 2 \quad (3.9)$$

#### 4–Heat and wave kernels for the Harmonic oscillator

The Cauchy problem  $(HL^a)$  for heat equation associated to the Harmonic oscillator for  $u_0 \in L^2(\mathbb{R})$  has the unique solution  $u$  belonging to  $C^0([0, \infty[, L^2(\mathbb{R}))$  see [6] more precisely there exists a semigroup  $(S_t)_{t \geq 0}$  of  $L^2(\mathbb{R})$ -contractions such that for all  $t > 0$ ,  $u(t, \cdot) = S_t u_0$ , the explicit solution given by

$$u(t, x) = \int_{-\infty}^{+\infty} K_a(t, x, x') u_0(x') dx'$$

where the kernel  $K_a(t, x, x')$  is given by the Mehler's formula (1.3)

**Theorem 4.1** If  $T$  is the intertwining operator given by (2.9) and if  $L^a$  and  $D$  are the harmonic oscillator and the Dirac operator given respectively by (1.1) and (1.2) then we have:

$$e^{tL^a} u_0 = T^{-1} \{e^{tD}(T u_0)\} \quad (4.1)$$

$$\frac{\sin t\sqrt{L^a}}{\sqrt{L^a}} v_0 = T^{-1} \left\{ \frac{\sin t\sqrt{D}}{\sqrt{D}} (T v_0) \right\} \quad (4.2)$$

where  $e^{tA}$  and  $\frac{\sin t\sqrt{A}}{\sqrt{A}}$  are respectively the heat and the wave kernels for the operator  $A$ .

**Proof** The formula (4.1) is a consequence of the formulas (2.10) and the Cauchy problems  $(HD)$  and  $(HL^a)$ .

The formula (4.2) is a consequence of the formulas (2.10) and the Cauchy problems  $(WL^a)$  and  $(WD)$ .

**Theorem 4.2** The Cauchy problem  $(HL^a)$  for the heat equation associated to the harmonic oscillator has the unique solution given by:

$$u(t, x) = \int_{-\infty}^{+\infty} H_a(t, x, x') u_0(x') dx' \quad (4.3)$$

where

$$H_a(t, x, x') =$$

$$a \sqrt{\frac{2}{\pi}} (e^{2at} - e^{-2at})^{-1/2} \exp \left\{ \frac{(e^{at}x - e^{-at}x')^2}{e^{2at} - e^{-2at}} + \frac{a}{2}(x^2 - x'^2) \right\} \quad (4.4)$$

**Proof:** Using (4.1) of theorem 4.1 we have

$$u(t, x) = e^{-at} e^{ax^2/2} \mathcal{F}^{-1} [e^{-(1-e^{-4at})\xi^2/4a} \mathcal{F} (e^{-ax^2/2} u_0) (\xi e^{-2at})] (x) \quad (4.5)$$

$$u(t, x) = \frac{1}{\sqrt{2\pi}} e^{-at} e^{ax^2/2} \times$$

$$\mathcal{F}^{-1} [e^{-(1-e^{-4at})\xi^2/4a}] * \mathcal{F}^{-1} [\mathcal{F} (e^{-ax^2/2} u_0) (\xi e^{-2at})] (x) \quad (4.6)$$

using (1.7) and (1.8) we can write

$$u(t, x) = \frac{e^{ax^2/2}}{\sqrt{\pi}} \left[ \frac{\sqrt{a}}{\sqrt{1-e^{-4at}}} e^{-\frac{ax^2}{1-e^{-4at}}} \right] * e^{2at} u_0(xe^{2at}) e^{-(a/2)x^2 e^{4at}} \quad (4.7)$$

$$= \frac{e^{at+ax^2/2}}{\sqrt{\pi}} \frac{\sqrt{a}}{\sqrt{1-e^{-4at}}} \int_{-\infty}^{+\infty} u_0(x' e^{2at}) e^{-(a/2)x'^2 e^{4at}} e^{\frac{-a(x-x')^2}{1-e^{-4at}}} dx' \quad (4.8)$$

set  $x'' = x' e^{2at}$  in (4.8) we get the formula (4.3).

**Remark 4.3:** The formula in (4.3) for the heat kernel associated to the harmonic oscillator agree with that given by (1.3) and the proof is left to the reader.

**Theorem 4.4** The Cauchy problem for the wave equation associated to the harmonic oscillator  $(WH)$  has the unique solution given by:

$$v(t, x) = -\frac{1}{a\sqrt{\pi}} e^{ax^2/2} \mathcal{F}^{-1} \left[ \frac{e^{-\xi^2/4a}}{\sqrt{\xi}} \int_{|\ln \xi/\xi'| < at} \operatorname{Erfc} \left( \frac{\sqrt{at}}{\sqrt{2|\ln \xi/\xi'|}} \right) \times \right. \\ \left. \frac{e^{\xi'^2/4a}}{\sqrt{\xi'}} \mathcal{F}(e^{ax'^2/2} v_0)(\xi') d\xi' \right] (x) \quad (4.9)$$

**Proof :** The formula (4.8) follows from the theorem (4.1) and the Fubini theorem using the formulas (3.2) and (3.2)' and the fact that

$$|\operatorname{Erfc}(z)| \leq \sqrt{\pi}$$

We finish this section by the following corollary:

**Corollary 4.5** The heat and wave kernels for the cursin operator

$$M = \frac{\partial^2}{\partial x^2} + x^2 \frac{\partial^2}{\partial y^2}$$

are given respectively by:

$$\begin{aligned}\tilde{H}_a(t, x, y, x', y', t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(y-y')a} H_a(t, x, x') da \\ \tilde{w}_a(t, x, y, x', y', t) &= \frac{1}{2\pi} \int_{-\infty}^{+\infty} e^{i(y-y')a} w_a(t, x, x') da\end{aligned}$$

### 5-Directions for further studies

We suggest here a certain number of open related problems connected to this paper. we are interested in the heat and wave equations for the harmonic oscillator with an inverse square potential

$$\begin{aligned}(HL^a)' \quad & \begin{cases} \partial_t u(t, x) = \left( \frac{\partial^2}{\partial x^2} - a^2 x^2 - \frac{b^2}{x^2} \right) u(t, x) & (t, x) \in \mathbb{R} \times \mathbb{R} \\ u(0, x) = u_0(x) & u_0 \in C_0^\infty(\mathbb{R}) \end{cases} \\ (WL^a)' \quad & \begin{cases} \partial_t^2 u(t, x) = \left( \frac{\partial^2}{\partial x^2} - a^2 x^2 - \frac{b^2}{x^2} \right) u(t, x) & (t, x) \in \mathbb{R}_+^* \times \mathbb{R} \\ u(0, x) = 0 & \partial_t u(0, x) = u_0(x) \in C_0^\infty(\mathbb{R}) \end{cases}\end{aligned}$$

Another possible extension is to consider the heat and wave equation associated to the power of the harmonic oscillator

$$\begin{aligned}(HL^a)'' \quad & \begin{cases} \partial_t u(t, x) = \left( \frac{\partial^2}{\partial x^2} - a^2 x^2 \right)^s u(t, x) & (t, x) \in \mathbb{R}_+^* \times \mathbb{R} \\ u(0, x) = u_0(x) & u_0 \in C_0^\infty(\mathbb{R}) \end{cases} \\ (WL^a)'' \quad & \begin{cases} \partial_t^2 u(t, x) = \left( \frac{\partial^2}{\partial x^2} - a^2 x^2 \right)^s u(t, x) & (t, x) \in \mathbb{R}_+^* \times \mathbb{R} \\ u(0, x) = 0 & \partial_t u(0, x) = u_0(x) \in C_0^\infty(\mathbb{R}) \end{cases}\end{aligned}$$

Finally, we suggest a problems in direction of the non linear heat and wave equations for the harmonic oscillator and to look for global solution and a possible blow up in finite times.

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